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One-Dimensional Caricature of Phase Transition

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In the limit as the volume grows and the temperature vanishes, it is shown that the one-dimensional nearest neighbor ferromagnetic Ising model presents a sharp transition between two different regimes. Fluctuations are studied in one of these regimes and also in the critical case.

KEY WORDS: Ising model; zero-temperature limit; sharp transition; fluctuations; criticality.

1. INTRODUCTION

We show in this paper that the very simple one-dimensional nearest neighbor ferromagnetic Ising model presents a sharp transition between two different regimes if one considers the limit in which the volume grows to infinity and the temperature vanishes simultaneously. The transition occurs with respect to a parameter which defines the relationship between he volume and the temperature.

Our motivation for considering such a subject comes from two sources. One is the fact that a similar phenomenon seems to occur in the much more elaborate model known as bootstrap percolation on the square lattice Z^2.[1,2] For this model if one lets the linear size on the system L grow to infinity as the parameter p (the initial density) goes to zero, keeping a relation of the type

$$L = \exp(\alpha/p)$$

then the asymptotic behavior has been proven to be different in the cases $\alpha < \alpha_1$ and $\alpha > \alpha_2$, where $0 < \alpha_1 < \alpha_2 < \infty$ are fixed constants. In fact, one

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expects a sharp threshold for α, separating the two regimes. One may argue that this soprt of transition may have some physical significance yielding an apparent critical point at finite volume which varies slowly with L. This in part may explain why for other related models simulations have indicated the presence of nontrivial critical points at infinite volume, contrary to rigorous results (see refs. 3, 7, and 8 and references given therein).

Another source of motivation is the research on metastability for Ising models with Glauber-type dynamics, in the limit as the temperature goes to zero. In ref. 5 this problem was investigated in the two-dimensional case for fixed volume. Related questions may be raised in the limit as the volume grows to infinity and the temperature vanishes. Here we start investigating this limit for the simpler one-dimensional Ising model in equilibrium. We consider a chain of N spins $\sigma = (\sigma_1, \ldots, \sigma_N)$ taking values -1 or $+1$ and interacting via the energy (free boundary conditions)

$$H_N(\sigma) = -(1/2) \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$$

The corresponding Gibbs measure at inverse temperature β is given by

$$\mu(\sigma) = Z^{-1} \exp\{-\beta H_N(\sigma)\}$$

where

$$Z = \sum_{\sigma \in \{-1, +1\}^N} \exp\{-\beta H_N(\sigma)\}$$

As is well known, this probability distribution corresponds to a Markov chain indexed by the sites $1, \ldots, N$, with transition probabilities given by

$$\mu(\sigma_{i+1} = -1/\sigma_i = -1) = 1 - p$$
$$\mu(\sigma_{i+1} = +1/\sigma_i = -1) = p$$
$$\mu(\sigma_{i+1} = -1/\sigma_i = +1) = p$$
$$\mu(\sigma_{i+1} = +1/\sigma_i = +1) = 2 - p$$

where

$$p = \frac{\exp(-\beta)}{1 + \exp(-\beta)}$$

The first spin, σ_1, takes the values $+1$ or -1 with equal probabilities 1/2.

We are interested in the limit when $\beta \to \infty$ and N is taken as

$$N = N(\beta) = [\exp(\alpha\beta)]$$

(where $[\cdot]$ denotes the integer part and $\alpha > 0$), which goes also to infinity. This means we are considering the asymptotic behavior of large systems at low temperature.

Let

$$m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i$$

be the average spin. The symbol δ_a will denote the measure concentrated on the number a, while $\to^D$ will denote convergence in distribution.

Proposition 1. (a) If $0 < \alpha < 1$, then $m \to^D \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$.

(b) If $\alpha > 1$, then $m \to^D \delta_0$.

Hence, there is a sharp threshold at $\alpha = 1$. In order to describe what happens at the critical value $\alpha = 1$, we need some definitions first. Let $X_k^1, \ldots, X_k^k$ be k independent random variables uniformly distributed between 0 and 1. Let $Y_k^1, \ldots, Y_k^k$ be their order statistics, i.e.,

$$Y_k^1 = \min\{X_k^l : l = 1, \ldots, k\}$$

and

$$Y_k^r = \min\{X_k^l : l = 1, \ldots, k, X_k^l > Y_k^{r-1}\}, \qquad r = 2, \ldots, k$$

Define the probability distributions μ_k, $k = 1, 2, \ldots$, on $\mathbb{R}$ by

$$\begin{aligned}\mu_k((-\infty, x]) = {} & \tfrac{1}{2}P((Y_k^1) - (Y_k^2 - Y_k^1) + (Y_k^3 - Y_k^2) - \cdots \\ & + (-1)^{k-1}(Y_k^k - Y_k^{k-1}) + (-1)^k(1 - Y_k^k) \leqslant x) \\ & + \tfrac{1}{2}P(-(Y_k^1) + (Y_k^2 - Y_k^1) - (Y_k^3 - Y_k^2) + \cdots \\ & + (-1)^k(Y_k^k - Y_k^{k-1}) + (-1)^{k+1}(1 - Y_k^k) \leqslant x)\end{aligned}$$

For each k, μ_k is clearly concentrated on $[-1, +1]$ and absolutely continuous with respect to the Lebesgue measure.

Proposition 2. If $\alpha = 1$, then

$$m \xrightarrow{D} e^{-1}(\tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_{+1}) + \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} \mu_k$$

Notice that in this case the limiting distribution has a discrete and an absolutely continuous part.

Part (b) of Proposition 1 suggests one should study the fluctuations of m around zero, in the limit. These fluctuations turn out to be normally distributed if m is properly rescaled. $N(0, 1)$ below denotes the standard normal distribution.

Proposition 3. If $\alpha > 1$, then

$$mN^{1/2}\{1/2\alpha \xrightarrow{D} N(0, 1)$$

Note that the exponent $1/2 - 1/2\alpha$ is close to 0 for α close to 1 and approaches the classical value 1/2 as $\alpha \to \infty$. As stated in Proposition 2, at $\alpha = 1$ the fluctuations are already present without rescaling and they are not normally distributed.

The proof of part (a) of Proposition 1 is trivial, since in the case $0 < \alpha < 1$ the probability that all spins have the same sign converges to one. Part (b) of Proposition 1 and Proposition 3 follow from the fact that if $\alpha > 1$, the number of boundaries between $+1$ and -1 spins grows to infinity, so that m is the average over many blocks of spins with opposite signs. We will prove Proposition 3 in Section 3. Part (b) of Proposition 1 is then a corollary. If $\alpha = 1$, the number of boundaries between spins $+1$ and -1 converges to a Poisson distribution, the positions of the boundaries being distributed as the order statistics described above. This is the reason for the behavior described in Proposition 2. The technical details are left to Section 2.

2. PROOF OF PROPOSITION 2

For $i = 1,..., N-1$, set

$$\eta_i = \sigma_i \sigma_{i+1}$$

The random variables η_i are independent, with a common distribution given by

$$P(\eta_i = -1) = p = 1 - P(\eta_i = +1)$$

Let K be the number of indices i for which $\eta_i = -1$. Then K is the number of successes in $N-1$ trials, each one with probability p of success. Therefore, since $\alpha = 1$,

$$\lim_{\beta \to \infty} P(K = k) = e^{-1}/k! \tag{2.1}$$

for $k = 0, 1, 2,...$. For fixed β, conditioned on $\{K = k\}$, the k indices i: $I^1_{\beta k} < I^2_{\beta k} < \cdots < L^k_{\beta k}$ for which $\eta_i = -1$ have their joint distribution given by the following construction: Take $J^1_{\beta k}$ uniformly from the set of indices $\{1,..., N-1\}$. Then take $J^2_{\beta k}$ uniformly from remaining set of indices, excluding $J^1_{\beta k}$. Proceed in this fashion by taking $J^l_{\beta k}$ uniformly from the remaining indices after excluding $J^1_{\beta k},\dots, J^{l-1}_{\beta k}$. Once we have $J^1_{\beta k},\dots, J^k_{\beta k}$, the random variables $I^1_{\beta k} < I^2_{\beta k} < \cdots < I^k_{\beta k}$ aıre their order statistics, i.e.,

$$I^1_{\beta k} = \min\{J^1_{\beta k} : l = 1,\dots, k\}$$

$$I^r_{\beta k} = \min\{J^l_{\beta k} : l = 1,\dots, k, J^l_{\beta k} > I^{r-1}_{\beta k}\} \qquad \text{for} \quad r = 2,\dots, k$$

It is also easy to see that for $x \in \mathbb{R}$ and $k = 1, 2,\dots$,

$$\begin{aligned}
&P(m \leqslant x \mid K = k)\\
&\quad = \tfrac{1}{2} P(N^{-1}[(I^1_{\beta k}) - (I^2_{\beta k} - I^1_{\beta k}) + (I^3_{\beta k} - I^2_{\beta k}) - \cdots\\
&\qquad + (-1)^{k-1}(I^k_{\beta k} - I^{k-1}_{\beta k}) + (-1)^k (N - I^k_{\beta k})] \leqslant x)\\
&\qquad + \tfrac{1}{2} P(N^{-1}[-(I^1_{\beta k}) + (I^2_{\beta k} - I^1_{\beta k}) - (I^3_{\beta k} - I^2_{\beta k}) + \cdots\\
&\qquad + (-1)^k (I^k_{\beta k} - I^{k-1}_{\beta k}) + (-1)^{k+1} (N - I^k_{\beta k})] \leqslant x)\\
&\quad = \tfrac{1}{2} P(f_k(I^1_{\beta k}/N,\dots, I^k_{\beta k}/N) \leqslant x)\\
&\qquad + \tfrac{1}{2} P(g_k(I^1_{\beta k}/N,\dots, I^k_{\beta k}/N) \leqslant x)
\end{aligned}$$

where

$$f_k(a^1,\dots, a^k) = a^1 - (a^2 - a^1) + \cdots + (-1)^{k-1}(a^k - a^{k-1}) + (-1)^k (1 - a^k)$$

and

$$g_k(a^1,\dots, a^k) = -a^1 + (a^2 - a^1) - \cdots + (-1)^k (a^k - a^{k-1}) + (-1)^{k+1}(1 - a^k)$$

As $\beta \to \infty$, the joint distribution of $(I^1_{\beta k}/N,\dots, I^k_{\beta k}/N)$ converges to that of $(Y^1_k,\dots, Y^k_k)$ defined in the introduction. Since f_k and g_k are continuous functions from $\mathbb{R}^k$ to $\mathbb{R}$, we have

$$\lim_{\beta \to \infty} P(m \leqslant x \mid K = k) = \tfrac{1}{2} P(f_k(Y^1_k,\dots, Y^k_k) \leqslant x) + \tfrac{1}{2} P(g_k(Y^1_k,\dots, Y^k_k) \leqslant x) \tag{2.2}$$

Now, using (2.1), (2.2), and Proposition 18, Section 11-4, p. 232, of ref. 6, we obtain

$$\begin{aligned}
&\lim_{\beta \to \infty} P(m \leqslant x)\\
&\quad = \lim_{\beta \to \infty} \sum_{k=0}^{\infty} P(m \leqslant x \mid K = k)\, P(K = k)\\
&\quad = e^{-1}\{\tfrac{1}{2} \mathbf{1}_{[-1,\infty]}(x) + \tfrac{1}{2} \mathbf{1}_{[+1,\infty]}(x)\}\\
&\qquad + \sum_{k=1}^{\infty} (e^{-1}/k!)\{\tfrac{1}{2} P(f_k(Y^1_k,\dots, Y^k_k) \leqslant x) + \tfrac{1}{2} P(g_k(Y^1_k,\dots, Y^k_k) \leqslant x)\}
\end{aligned}$$

as we wanted to show.

3. PROOF OF PROPOSITION 3

For convenience, we will embed the N spins in a semi-infinite Markov chain $\sigma_1, \sigma_2,\ldots$ with

$$P(\sigma_i = -1) = P(\sigma_i = +1) = 1/2$$

and the same transition matrix mentioned in the introduction. We look to the blocks of spins $+1$ and the blocks of spins -1. As in the last section, set $\eta_i = \sigma_i \sigma_{i+1}$ and define

$$I^1 = \min\{i = 1, 2,\ldots: \eta_i = -1\}$$
$$I^k = \min\{i = 1, 2,\ldots: \eta_i = -1, i > I^{k-1}\}, \qquad k = 2, 3,\ldots$$

X_1 and Y_1 will denote respectively the lengths of the lth blocks of spins $+1$ and of spins -1. This means that on the event $\{\sigma_1 = +1\}$ we set

$$X_1 = I^1$$
$$X_1 = I^{2l} - 1 - I^{2l-2X}, \qquad l = 2, 3,\ldots$$
$$Y_l = I^{2l} - I^{2l} - 1, \qquad l = 1, 2, 3,\ldots$$

and on the event $\{\sigma_1 = -1\}$ we interchange the definitions of X_l and Y_l above. Let also

$$Z_l = X_l + Y_l$$

be the length of the lth pair of blocks of spins $+1$ and -1. For fixed β the random variables $X_1, X_2,\ldots$ and $Y_1, Y_2,\ldots$ are all mutually independent and have the common geometric distribution

$$P(X_l = k) = P(Y_1 = k) = (1-p)^{k-1} p, \qquad k = 1, 2,\ldots$$

We set

$$l_0 = N/E(Z_1) = (p/2)N \sim (1/2)\exp\{(\alpha - 1)\beta\}$$

where $R(\cdot)$ denotes expectation and $A \sim B$ means that the ratio A/B approaches 1 as $\beta \to \infty$. Now we define

$$N_0 = \sum_{l=1}^{l_0} Z_l$$

Observe that $E(N_0) = N$. We will show that

$$\frac{\sum_{i=1}^{N_0} \sigma_i}{N^{1/2 + 1/2\alpha}} \xrightarrow{D} N(0, 1) \qquad (3.1)$$

and we will finish the proof of Proposition 3 later by showing that

$$\frac{|\sum_{i=1}^{N} \sigma_i - \sum_{i=1}^{N_0} \sigma_i|}{N^{1/2+1/2\alpha}} \xrightarrow{D} \delta_0 \tag{3.2}$$

To prove (3.1), we write

$$\sum_{i=1}^{N_0} \sigma_i = \sum_{l=1}^{l_0} (X_l - Y_l) \tag{3.3}$$

and use a central limit theorem for triangular arrays from ref. 4 (Theorem 1, Section 49, p. 283). There is a minor technical detail we have to note: we are letting the continuous parameter β go to infinity, while the theorem we want to apply is stated for a discrete parameter. This is easy to overcome by observing that we can in fact consider that we are taking a generic sequence $\{\beta_i\}$ of values of β such that $\beta_i \to \infty$ as $i \to \infty$. With this in mind and using (3.3), we write

$$\xi_{\beta l} = (X_l - Y_l)/N^{1/2+1/2\alpha}$$

and

$$\zeta_\beta = \sum_{l=1}^{l_0} \xi_{\beta l} = \left(\sum_{i=1}^{N_0} \sigma_i\right) \Big/ N^{1/2+1/2\alpha}$$

Theorem 1, p. 283 of ref. 4 states that (3.1) follows once we show that:

(i) The variables $\xi_{\beta l}$ have finite variances.

(ii) The variances of the sums ξ_β are bounded by a constant C not dependent on β.

(iii) $\mathrm{Var}(\xi_{\beta 1})$ goes to zero as $\beta \to \infty$, where $\mathrm{Var}(\cdot)$ denotes the variance.

(iv) $l_0 E(\xi_{\beta 1}) \to 0$ as $\beta \to \infty$.

(v) For every $\tau > 0$,

$$l_0 \int_{|x| \neq \tau} x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) \leqslant x) \to 0 \qquad \text{as} \quad \beta \to \infty$$

(vi) For every $\tau > 0$,

$$l_0 \int_{|x| < \tau} x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) \leqslant x) \to 1 \qquad \text{as} \quad \beta \to \infty$$

Conditions (i)–(iii) are the conditions for the random variables $\{\xi_{\beta l}\}$ to form an "elementary system" in the terminology of ref. 4 (see Section 47, p. 278). Conditions (iii)–(vi) are simpler than the corresponding statements in ref. 4 because in our case we have translation invariance for each fixed β.

Conditions (i)–(iii) are clearly true since

$$\operatorname{Var}(\xi_{\beta l}) = \frac{\operatorname{Var}(X_l) + \operatorname{Var}(Y_l)}{N^{1+1/\alpha}} = \frac{2(1-p)}{p^2 N^{1+1/\alpha}} \sim \frac{2}{e^{\beta(\alpha-1)}}$$

and

$$\operatorname{Var}(\zeta_\beta) = l_0 \operatorname{Var}(\xi_{\beta 1}) \sim 1$$

Condition (iv) is trivial since by symmetry $E(\xi_{\beta 1}) = \Lambda 0$. To prove (v), we write

$$\int_{|x|>\tau} x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) \leqslant x) \leqslant \sum_{j=1}^{\infty} \{\tau(j+1)\}^2 \, P(|X_1 - Y_1|/N^{1/2+1/2} > j\tau)$$

But

$$\begin{aligned} &P(|X_1 - Y_1|/N^{1/2+1/2\alpha} > j\tau) \\ &\quad \leqslant 2P(X_l > j\tau N^{1/2+1/2\alpha}) \\ &\quad = 2(1-p)\, j\tau N^{1/2+1/2\alpha} \\ &\quad \leqslant 2\exp(j\tau N^{1/2+1/2\alpha} p) \end{aligned}$$

Now note that $N^{1/2+1/2\nu} p \sim \exp\{\beta(\alpha-1)/2\}$, which diverges as $\beta \to \infty$. Hence, for large β,

$$r := \exp(-\tau N^{1/2+1/2\alpha} p) < 1$$

and then

$$\int_{|x|>\tau} x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) \leqslant x) \leqslant 2 \sum_{j=1}^{\infty} \{\tau(j+1)\}|^2 r^j \leqslant Cr$$

where C is a finite constant which does not depend on β, for large β. From the above we see that

$$\int_{|x|>\tau} x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) \leqslant x)$$

goes to zero as $\beta \to \infty$ faster than any exponential of β, while l_0 goes to infinity as the exponential $e^{(\alpha-1)\beta}$. This implies that (v) is satisfied.

Condition (vi) follows from (v) and the fact that

$$\begin{aligned} l_0 \int x^2 \, dP(\xi_{\beta 1} - E(\xi_{\beta 1}) > x) &= l_0 E(\xi_{\beta 1})^2 \\ &= l_0 \{E(X_1)^2 + E(Y_1)^2 - 2E(X_1)\, E(Y_1)\}\, N^{-1-1/\alpha} \\ &= l_0 \left(2 \frac{1-p}{p^2}\right) N^{-1-1/\alpha} \sim 1 \end{aligned}$$

We turn now to the proof of (3.2). First, we note that by defining

$$\bar{\xi}_{\beta l} = \{X_l + Y_l - E(X_l) - E(Y_l)\}/N^{1/2 + 1/2\alpha}$$

and

$$\bar{\zeta}_\beta = \sum_{l=1}^{l_0} \bar{\xi}_{\beta l}$$

one can prove as above that

$$\frac{N_0 - N}{N^{1/2 + 1/2\alpha}} = \bar{\zeta}_\beta \xrightarrow{D} N(0, 1) \qquad \text{as} \quad \beta \to \infty$$

But $N^{1/2 + 1/2\alpha} \sim \exp\{\beta(1+\alpha)/2\}$, so that for every $\delta > 0$,

$$\lim_{\beta \to \infty} P\left(|N_0 - N| > \exp\left\{\left(\frac{1+\alpha}{2} + \delta\right)\beta\right\}\right) = 0 \tag{3.4}$$

Set

$$\bar{N} = \left[\exp\left\{\left(\frac{1+\alpha}{2} + \delta\right)\beta\right\}\right]$$

and consider the sites

$$A = N - \bar{N}$$

$$B = N + \bar{N}$$

Now

$$\begin{aligned} P&\left(\frac{|\sum_{i=1}^{N} \sigma_i - \sum_{i=1}^{N_0} \sigma_i|}{N^{1/2 + 1/2\alpha}} > \varepsilon\right) \\ &\leqslant \left(|N_0 - N| > \exp\left\{\left(\frac{1+\alpha}{2} + \delta\right)\beta\right\}\right) \\ &\quad + P\left(\max_{j = A,\dots,N)} \left|\frac{\sum_{i=j}^{N} \sigma_i}{N^{1/2 + 1/2\alpha}}\right| > \varepsilon\right) + P\left(\max_{j = N,\dots,B)} \left|\frac{\sum_{j=N}^{j} \sigma_i}{N^{1/2 + 1/2\alpha}}\right| > \varepsilon\right) \end{aligned} \tag{3.5}$$

The last two terms are obviously identical. We will control the latter one. First we introduce blocks of spin $+1$ and -1 in thne same fashion as we did before, but this time starting from the site N and going to the right.

Let U_l be the length of the lth block of $+1$ spins and V_1 be the length of the lth block of -1 spins. Set also

$$K = \max\{l: (U_1 + V_1) + (U_2 + V_2) + \cdots + (U_l + V_l) < \bar{N}\}$$

and

$$k = \left[\exp\left\{\left(\frac{\alpha - 1}{2} + 2\delta\right)\beta\right\}\right]$$

Then

$$P\left(\max_{j = N,\dots,B}\left|\frac{\sum_{i=N}^{j}\sigma_i}{N^{1/2+1/2\alpha}}\right| > \varepsilon\right)$$

$$\leqslant P(K > k) + P\left(\max_{s=1,\dots,k}\left|\frac{U_i}{N^{1/2+1/2\alpha}}\right| > \varepsilon/2\right)$$

$$P\left(\max_{i=1,\dots,k+1}\left|\frac{V_i}{N^{1/2+1/2\alpha}}\right| > \varepsilon/2\right) \tag{3.6}$$

To control the first term on the rhs of (3.6), we observe that

$$\frac{\sum_{i=1}^{k}\{(U_i + V_i) - E(U_i + V_i)\}}{kE(U_1 + V_1)} \xrightarrow{D} \delta_0 \tag{3.7}$$

To prove (3.7), one can show the stronger statement

$$\frac{\sum_{i=1}^{k}\{(U_i + V_i) - E(U_i + V_i)\}}{2^{1/2}\exp\{[(\alpha+3)/4 + \delta]\beta\}} \xrightarrow{D} N(0, 1) \qquad \text{as} \quad \beta \to \infty \tag{3.8}$$

(3.8) can be proven by the same method used to prove (3.1). It implies (3.7) because

$$k(E(U_1 + V_1)) \sim \exp\left\{\left(\frac{\alpha + 1}{2} + 2\delta\right)\beta\right\} \tag{3.9}$$

and $(\alpha + 1)/2 > (\alpha + \; + 3)/4$, Now using (3.7) and (3.9), we have

$$\limsup_{\beta\to\infty} P(K > k) \leqslant \limsup_{\beta\to\infty} P\left(\sum_{i=1}^{k}(U_i + V_i) < \bar{N}\right) = 0 \tag{3.10}$$

The second term on the rhs of (3.6) can be controled using Kolmogorov's inequality (see ref. 4, Section 34, p. 213):

$$P\left(\max_{s=1,\dots,k} \left|\frac{\sum_{i=1}^{s}(U_i - V_i)}{N^{1/2+1/2\alpha}}\right| > \varepsilon/2\right) \leqslant 4\,\frac{k\,\mathrm{Var}(U_1 - V_1)}{\varepsilon^2 N^{1+1/\alpha}}$$

$$\sim \frac{8}{\varepsilon^2}\exp\left\{\left(\frac{1-\alpha}{2} + 2\delta\right)\beta\right\} \qquad (3.11)$$

which vanishes as $\beta \to \infty$, provided we have chosen

$$0 < \delta < \frac{\alpha - 1}{4} \qquad (3.12)$$

Finally, the last two terms on the rhs of (3.6) are identical and are easily controlled:

$$P\left(\max_{i=1,\dots,k+1} \left|\frac{U_i}{N^{1/2+1/2\alpha}}\right| > \frac{\varepsilon}{2}\right)$$

$$\leqslant kP\left(U_1 > \frac{\varepsilon}{2} N^{1/2+1/2\alpha}\right)$$

$$= k(1-p)^{[(\varepsilon/2)N^{1/2+1/2\alpha}]}$$

$$\leqslant \exp\left\{\left(\frac{\alpha-1}{2} + 2\delta\right)\beta\right\}\exp\left(-p\left[\frac{\varepsilon}{2}\exp\left\{\frac{1}{2}\beta(\alpha+1)\right\}\right]\right) \qquad (3.13)$$

which vanishes as $\beta \to \infty$.

Now it is easy to see that (3.2) follows from (3.4)–(3.6) and (3.10)–(3.13).

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